MATH 6021 Lecture 4 9/28/2020

<u>Recall</u>: $\Sigma^k \subseteq (M^n, g)$ min $\Longleftrightarrow$ $\vec{H} \equiv 0$.

$$\delta\Sigma(X) = \int_\Sigma \mathrm{div}_\Sigma X \, dV = -\int_\Sigma \langle \vec{H}, X \rangle \, dV$$

$$\rightsquigarrow \delta^2\Sigma(X) = \int_\Sigma \left(|\nabla\varphi|^2 - (\mathrm{Ric}^M(\nu,\nu) + |A|^2)\varphi^2 \right) dV.$$

$k=n-1$

2-sided

$X = \varphi\nu$

<u>Def^n</u>: Σ stable $\Longleftrightarrow$ $\delta^2\Sigma(X) \geq 0$ $\forall X$ cpt. supp.

Now, $M^n = \mathbb{R}^n$.

<u>Bernstein</u>: $\Sigma^2 \subseteq \mathbb{R}^3$ entire min. graph $\Rightarrow$ flat

(graph $\Uparrow$ area-minimizing)

<u>Fisher-Colbrie-Schoen / Do Carmo-Peng</u>: $\Sigma^2 \subseteq \mathbb{R}^3$ complete, stable min. $\Rightarrow$ flat

<u>Stable Bernstein Conj</u>: $\Sigma^{n-1} \subseteq \mathbb{R}^n$ complete, stable $n \leq 7$ $\Rightarrow$ flat.

<u>Schoen-Simon-Yau '75</u>: True, if assume Euclidean volume growth.

<u>Note</u>: The proof of curvature estimates for any <u>immersed</u> min. hypersurfaces rely on a useful differential inequality, known as "Simons inequality":

$\Sigma^{n-1} \subseteq M^n$ min. hypersurf. (i.e. $H \equiv 0$) $\Rightarrow$ $$\boxed{\Delta_\Sigma |A|^2 \geq -C(1+|A|^2)^2}$$

where $C > 0$ is a constant depending on n and the geometry of M.

<u>Remark</u>: (1) pointwise inequality.

(2) no stability / 2-sided assumption

We will give a proof of the case in $\mathbb{R}^n$.

Simons Inequality in $\mathbb{R}^n$: Let $\Sigma^{n-1} \subset \mathbb{R}^n$ be an immersed min hypersurf.

THEN,

$$\Delta_\Sigma |A|^2 \geq -2|A|^4 + 2\left(1 + \frac{2}{n-1}\right) |\nabla_\Sigma |A||^2$$

[Note: $\Delta_\Sigma |A|^2 \geq -2|A|^4$

& $|A| \Delta_\Sigma |A| \geq -|A|^4 + \frac{2}{n-1} |\nabla_\Sigma |A||^2$]

(Sketch of) Proof:

Note: ptwise calculation. Fix $p \in \Sigma$, and a local O.N.B. $\underbrace{E_1, \dots, E_{n-1}}_{\text{tang. to } \Sigma}, \underbrace{E_n}_{\perp \Sigma}$

Write in this basis $A = (a_{ij})$ symm. $(0,2)$-tensor

Idea: Take many derivatives & switching their orders ...

Gauss eq^{n}: $R_{ijkl} = a_{ik} a_{jl} - a_{jk} a_{il}$, R = Riem. curv. of Σ

Codazzi eq^{n}: $a_{ij;k} = a_{ik;j} = a_{ji;k}$, i.e. $a_{ij;k}$ fully symm.

(Note: $|A|^2 = \sum_{i,j} a_{ij}^2$) By switching order of covariant derivatives

$$a_{ij;kl} = a_{ij;lk} + \sum_m R_{lkim} a_{mj} + \sum_m R_{lkjm} a_{mi}$$

We compute

$$\frac{1}{2} \Delta_\Sigma |A|^2 = \underbrace{\sum_{i,j} a_{ij} \Delta_\Sigma a_{ij}}_{\langle A, \Delta A \rangle} + \underbrace{\sum_{i,j} |\nabla_\Sigma a_{ij}|^2}_{|\nabla A|^2}$$

$$= \sum_{i,j,k} a_{ij} \underset{\substack{\| \text{ Codazzi} \\ a_{ik;jk}}}{a_{ij;kk}} + \sum_{i,j,k} a_{ij;k}^2$$

$$= \sum_{i,j,k} a_{ij} \Big(\underbrace{a_{ik;kj}}_{\substack{\| \text{ Codazzi} \\ a_{kk;ij} \\ = 0 \ \because \Sigma \text{ min.}}} + \sum_m \underbrace{R_{kjim} a_{mk}}_{(a_{ki} a_{jm} - a_{ji} a_{km}) a_{mk}} + \sum_m \underbrace{R_{kjkm} a_{mi}}_{(a_{kk} a_{jm} - a_{jk} a_{km}) a_{mi}} \Big) + \sum_{i,j,k} a_{ij;k}^2$$

Gauss

$(H = \sum_k a_{kk} = 0)$

cancels

0 =

cancels

$$= - \sum_{i,j,k,m} a_{ij}^2 a_{km}^2 + \sum_{i,j,k} a_{ij;k}^2$$

i.e. $$\boxed{\Delta_\Sigma |A|^2 = -2|A|^4 + 2|\nabla A|^2}$$ "Simons Identity".

Using "Enhanced Kato's ineq.", we have

$$|\nabla A|^2 \geq \left(1+\frac{2}{n-1}\right) |\nabla_\Sigma |A||^2 .$$

Remark: "=" holds in $n=3$.

Next, we combine Simons ineq. and stability ineq. to obtain higher L^p-bounds for $|A|^2$.

L^p-estimate of SSY '75

Let $\Sigma^{n-1} \subseteq \mathbb{R}^n$ be a 2-sided, stable min. hypersurface.
Then, $\forall\, p \in [2, 2+\sqrt{\frac{2}{n-1}})$, we have

$$\int_\Sigma |A|^{2p} \phi^{2p} \leq C(n,p) \int_\Sigma |\nabla \phi|^{2p} \qquad \forall\, \phi \in C_c^\infty(\Sigma).$$

Proof: Recall: Stability ineq. $\int |A|^2 \eta^2 \leq \int |\nabla \eta|^2 \quad \forall \eta \in C_c^\infty(\Sigma)$

Take $\eta := |A|^{1+q} f$ where $f \in C_c^\infty(\Sigma)$, and $q \in [0, \sqrt{\frac{2}{n-1}})$

$$\underbrace{\boxed{\int |A|^{4+2q} f^2}}_{(0)} \leq \int |f \nabla(|A|^{1+q}) + |A|^{1+q} \nabla f|^2$$

$$= (1+q)^2 \underbrace{\boxed{\int f^2 |A|^{2q} |\nabla |A||^2}}_{(\mathrm{I})} + \underbrace{\boxed{\int |A|^{2+2q} |\nabla f|^2}}_{(\mathrm{II})}$$

$$+ 2(1+q) \underbrace{\boxed{\int f |A|^{1+2q} (\nabla f \cdot \nabla |A|)}}_{(\mathrm{III})}$$

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Idea: Keep (II) and estimate / absorb (I) and (III)

Use Simons ineq: $|A| \Delta |A| + |A|^4 \geq \frac{2}{n-1} |\nabla |A||^2$ to estimate (I).

Multiply the ineq. by $|A|^{2q} f^2$ and integrate

$$\underbrace{\frac{2}{n-1} \int f^2 |A|^{2q} |\nabla |A||^2}_{(\mathrm{I})} \leq \int \underbrace{f^2 |A|^{1+2q} \Delta |A|}_{\text{int. by part: } \text{"} \operatorname{div}(f^2 |A|^{1+2q} \nabla |A|) - \nabla (f^2 |A|^{1+2q}) \cdot \nabla |A| \text{"}} + f^2 |A|^{4+2q}$$

$$= \underbrace{-2 \int f |A|^{1+2q} \nabla f \cdot \nabla |A|}_{(\mathrm{III})} - (1+2q) \underbrace{\int f^2 |A|^{2q} |\nabla |A||^2}_{(\mathrm{I})} + \underbrace{\int |A|^{4+2q} f^2}_{(0)} .$$

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Adding 1 & 2, we obtain

$$\underbrace{\left(\frac{2}{n-1} - q^2\right)}_{>0} \underbrace{\int f^2 |A|^{2q} |\nabla |A||^2}_{(\mathrm{I})} \leq \underbrace{\int |A|^{2+2q} |\nabla f|^2}_{(\mathrm{II})} + 2q \underbrace{\int f |A|^{1+2q} \nabla f \cdot \nabla |A|}_{(\mathrm{III})}$$

when $q^2 < \frac{2}{n-1}$ (*)

By "weighted" Cauchy-Schwarz : $2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2$,

$$2q \underbrace{\int f |A|^{1+2q} \nabla f \cdot \nabla |A|}_{(\mathrm{III})} \leq \varepsilon q \underbrace{\int f^2 |A|^{2q} |\nabla |A||^2}_{(\mathrm{I})} + \frac{q}{\varepsilon} \underbrace{\int |A|^{2+2q} |\nabla f|^2}_{(\mathrm{II})}$$

Choose $\varepsilon > 0$ small enough, dep on q, n. then.

$$\underbrace{\left(\frac{2}{n-1} - q^2 - \varepsilon q\right)}_{>0} \underbrace{\int f^2 |A|^{2q} |\nabla |A||^2}_{(\mathrm{I})} \leq \left(1 + \frac{q}{\varepsilon}\right) \underbrace{\int |A|^{2+2q} |\nabla f|^2}_{(\mathrm{II})}$$

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Using Cauchy-Schwarz in ① & ③, we have

$$\underbrace{\int |A|^{4+2q} f^2}_{(0)} \le \left(2 + \frac{2(1+q^2)(1+\frac{q}{\varepsilon})}{\frac{2}{n-1} - q^2 - \varepsilon q}\right) \underbrace{\int |A|^{2+2q} |\nabla f|^2}_{(\mathrm{II})}$$

Set $p = 2+q \in [2, 2+\sqrt{\frac{2}{n-1}})$, and $f = \phi^p$. Then,

$$\int |A|^{2p} \phi^{2p} \le C(n,p) \int |A|^{2p-2} \phi^{2p-2} |\nabla \phi|^2$$

Hölder ineq.

$$\le C(n,p) \underline{\left(\int |A|^{2p} \phi^{2p} \right)^{\frac{p-1}{p}}} \left(\int |\nabla \phi|^{2p} \right)^{\frac{1}{p}}$$

divide

∎

Cor: Assume (i) $\Sigma^{n-1} \subseteq \mathbb{R}^n$ complete, 2-sided, stable min. hypersurface

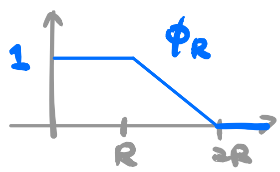
(ii) $\exists C > 0$ s.t. $|\Sigma \cap B_R| \le C R^{n-1}$ $\forall R > 0$.

intrinsic ball in Σ

(iii) $3 \le n \le 6$

Then, Σ is flat.

Proof: Take cutoff fcn $\phi = \phi_R =$



$|\nabla \phi_R| \approx \frac{1}{R}$

By L^p-estimate,

$$\int_{\Sigma \cap B_R} |A|^{2p} \le \int_\Sigma |A|^{2p} \phi_R^{2p} \le C \int_{\underbrace{\Sigma \cap B_{2R}}_{C \cdot (2R)^{n-1}}} \overbrace{|\nabla \phi_R|^{2p}}^{(\frac{1}{R})^{2p}} \le C' R^{-2p+n-1}$$

Note: if $\underbrace{-2p+n-1}_{\text{Recall: } p \in [2, 2+\sqrt{\frac{2}{n-1}})} < 0$, then done by taking $R \to \infty$.

Want: $p > \frac{n-1}{2}$.

So, need $\frac{n-1}{2} < 2 + \sqrt{\frac{2}{n-1}}$. ✓ holds $3 \le n \le 6$.

∎

Remark: (1) For $n=7$, L. Simon '76 for embedded.

For embedded case, different treatment by Schoen-Simon '81.

(2) By Moser iteration, improve L^p-bdd to L^∞-bdd for $|A|^2$ away from $\partial\Sigma$.

(3) 2D situation does not require a-prior area bdd.

(c.f. Schoen '83, Colding-Minicozzi '02)

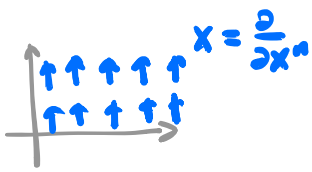
Q: What can we say if we do NOT assume stability?

Recall: $\Sigma^k \subseteq \mathbb{R}^n$ min (i.e. $\vec{H} \equiv 0$) $\iff$ $\delta\Sigma(X) := \int_\Sigma \mathrm{div}_\Sigma X = 0$

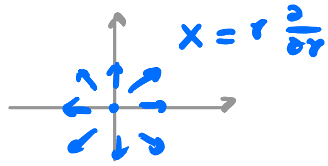
for all cpt supp. vector fields X in $\mathbb{R}^n$

Idea: "suitable" choice of X yields information about Σ.

Eg.) X = translation



X = scaling / dilation



Prop: $\Sigma^k \subseteq \mathbb{R}^n$ min. $\iff$ coordinate functions $x^1, \dots, x^n$ on $\mathbb{R}^n$ restrict to harmonic functions on Σ.

i.e. $\Delta_\Sigma x^i = 0$, for $i = 1, \dots, n$

Proof: Take $X = \eta \frac{\partial}{\partial x^i}$ where η is smooth, cpt supp.

$$\mathrm{div}_\Sigma X = \mathrm{div}_\Sigma\left(\eta \frac{\partial}{\partial x^i}\right) = \nabla^\Sigma \eta \cdot \left(\frac{\partial}{\partial x^i}\right)^T = \nabla^\Sigma \eta \cdot \nabla^\Sigma x^i$$

Apply 1st var. formula and Stokes' Thm,

$$0 = \int_\Sigma \operatorname{div}_\Sigma X = \int_\Sigma \nabla^\Sigma \eta \cdot \nabla^\Sigma x^i = -\int_\Sigma \eta \, \Delta^\Sigma x^i \quad \forall \eta \in C_c^\infty(\Sigma)$$

$$\Uparrow$$

$$\Delta^\Sigma x^i = 0.$$

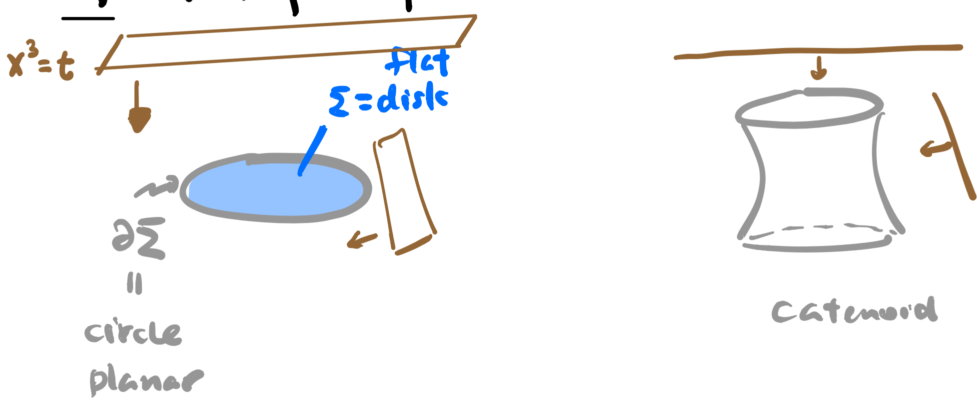
Cor: (Convex Hull Property)

Any compact min. submfd $\Sigma^k \subseteq \mathbb{R}^n$ is contained inside the "convex hull" of its boundary $\partial\Sigma$.

the smallest convex set containing $\partial\Sigma$.

In particular, $\nexists$ compact min. submfd in $\mathbb{R}^n$ without boundary.

"Pf": max. principle for harmonic functions.



Remark: This is special to the ambient space being $\mathbb{R}^n$.

One of the most important tool in studying min. surfaces in the following

Monotonicity Formula

Let $\Sigma^k \subseteq \mathbb{R}^n$ min. subfld, fix a pt $x_0 \in \mathbb{R}^n$ (not nec. in Σ).

Then, $\forall\, 0 < s < t < \operatorname{dist}_{\mathbb{R}^n}(x_0, \partial\Sigma)$

$$\frac{|\Sigma \cap B_t(x_0)|}{t^k} - \frac{|\Sigma \cap B_s(x_0)|}{s^k} = \int_{\Sigma \cap (B_t \setminus B_s)} \frac{|(x-x_0)^\perp|^2}{|x-x_0|^{k+2}} \geq 0$$

thus $t \longmapsto \dfrac{|\Sigma \cap B_t(x_0)|}{t^n}$ is non-decreasing.